Executive Summary

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The main focus of my research is the Partial Differential Equations (PDE) in mathematical biology and fluid mechanics. My goal is to give faithful descriptions of a broad range of interactions between biological phenomena and fluid motion. I am interested in the long-time behaviors of interacting bio-fluid systems, flocking phenomena of birds and fish, and small-scale creation and stability in fluid mechanics. These systems are challenging because the nonlinearities involved are usually non-local, which complicates the behavior of solutions and analysis.

1 Biological Systems in Fluid Streams: Suppression of Blow-up and Enhancement of Reaction

The first part of the my research focuses on the influence of fluid stream on various biological phenomena. The following equation establishes the basic framework:

$$\partial_t n + \underbrace{\mathbf{u}\cdot\nabla n}_{\text{fluid transport}} = \underbrace{\Delta n}_{\text{diffusion}} + \underbrace{\mathcal{NL}}_{\text{biological effect}} . \tag{1.1}$$

The equation models the time evolution of the biological substance subject to fluid transportation, random Brownian motion, and various types of biological effects. Here, the solution n denotes the biological substance's density, and the divergence-free vector field $\mathbf{u}$ represents the ambient fluid current. The nonlinear term $\mathcal{NL}$ models different biological interactions.

I consider two types of biological phenomena. The first one is the chemotaxis of micro-organism. In various circumstances, cells secret chemical signals to attract others of the same kind. The signal's gradient directs the aggregation of the cells. One expects that if the population is enormous, the cells form massive clusters. In the differential equation language, we say that blow-up formation occurs. I am interested in the delicate interaction between the cluster formation and the underlying fluid stream. The Patlak-Keller-Segel equations are widely applied to model the chemotaxis phenomena in the math community, see, e.g., [21], [17]. The second phenomenon is the fertilization of marine animals. In the ocean, marine animals such as abalones, corals, and shrimp release their eggs and sperms in the fluid stream. Once the gametes meet, the fertilization happens. The fluid stream and chemical attraction play crucial roles in fertilization success. We developed a two-species advection-reaction-diffusion system to analyze this fertilization process.

Together with E. Tadmor, J. Bedrossian and A. Zlatoš, I showed in the papers [2], [14], [8], [15] that the strong shear flow or hyperbolic flow has the potential of suppressing chemotactic blow-up in the Patlak-Keller-Segel systems. In the paper [10], together with A. Kiselev, I showed that the shear flow or stochastic flow accelerates the two-species reaction speed, which confirms the observation in the chemistry community.
Overall Significance The project enhances the current understanding of chemotaxis and relates two distinct areas in mathematics, namely, mathematical biology and fluid mechanics. Novel ideas and techniques from the study of the mixing phenomena and hydrodynamic stability ([1],[5]) found their new applications here. To the author's knowledge, [2] is the first work that uses hypocoercivity to obtain enhanced dissipation estimates for nonlinear problems. The paper [14] is the first result of the suppression of blow-up for PKS on the plane. In [15], we further extend it to include the general initial data. The paper [8] is the first result of the suppression of blow-up through fluid flow for the fully parabolic PKS equation. In the paper [7], we capture the critical mass threshold in the coupled Patlak-Keller-Segel-Navier-Stokes equations for the first time.

2 Hydrodynamic Flocking: Single and Multi-species Perspectives

Flocking behavior occurs when many agents (fish or birds) move as a group. In nature, starlings and sardines form large flocks or swarms to avoid predators. The alignment effect becomes apparent here because individual agent tends to adjust its velocity u to a weighted average of its neighbors ($\overline{u}$) to avoid running into others. The governing equation describes the evolution of the population density ρ, and the agent velocity u:

$$u_t + \overbrace{u \cdot \nabla u}^{\text{advection}} = \overbrace{\overline{u} - u}^{\text{alignment}}, \quad \rho_t + \nabla \cdot (\rho u) = 0. \tag{2.1a}$$

The weighted average velocity $\overline{u}$ is dynamically determined through the influence function ϕ:

$$\overline{u} = \frac{\phi * (\rho u)}{\phi * \rho}. \tag{2.1b}$$

The multi-species analog of the dynamics (2.1) is also of interest since it describes how different flocks interact. One can formally imagine that colors are given to species, and the mixing of different colors introduce rich dynamics.

My project is to study the global well-posedness theory of the equation (2.1) and its multi-species counterpart in dimension two. In the paper [13], to gether with E. Tadmor, I found critical threshold behavior (see, e.g., [6],[19],[20]) in the system and gave simple criterion to check global well-posedness. In the paper [12], we discover that the multi-species system's long-time behavior has a close relationship with the graph Laplacian of the interaction matrices.

Overall Significance By carefully studying the evolution of the symmetrized velocity gradient matrix's spectral gap and the vorticity, we gave a simple explicit criterion to determine well-posedness. This greatly enhances the current understanding of two-dimensional flocking dynamics. By introducing the multi-species concept into the flocking hydrodynamics [12], we establish connections between graph theory and PDE analysis, breaking new grounds for the current research on collective behavior.

3 Fluid Dynamics: Small Scale Creation and Boundary Layer Stability

I am interested in small-scale creation and hydrodynamics stability. Some results are related to the boundary layer of fluids.

The first problem is related to the hydrodynamics stability of the Navier-Stokes equation. In the recent years, stability of the Couette flow on the domain $\mathbb{T}\times\mathbb{R}$ has attracted a lot of attention due to its relation to the Landau damping effect, see, e.g., [1], [4], [5]. Together with Jacob Bedrossian, we consider the Couette flow's linear stability subject to the non-slip boundary condition in the channel $\mathbb{T}\times[-1,1]$.

The second problem is the exponential growth of the derivatives of the vorticity in the inviscid SQG equation. Together with A. Kiselev, we constructed exponentially growing solutions to the equation.

The third problem is about blow-up solutions to boundary layer models of the 2D Boussinesq equation and 3D Euler. In the paper [16], T. Hou and G. Luo proposed a potential blow-up scenario for the 3D Euler. Based on the Hou-Luo scenario, in the paper [18], A.Kiselev and V. Sverak proved sharp double exponential growth of the 2-dimensional Euler equation in a disk. Since the explicit construction of a smooth blow-up solution to the 3-dimensional Euler system is still out of reach, various models were proposed and analyzed by various authors. We consider several models [9] that capture the boundary layer effect and blow-up suppression mechanism in the dynamics.

Overall Significance When studying the stability of Couette flow in the channel, we observed that the vorticity undergo enhanced dissipation, [3]. Moreover, the boundary vorticity undergoes inviscid damping, which is not reported in the literature before. The result builds a bridge between the inviscid damping of the Couette flow and the boundary layer dynamics.

To our knowledge, the exponential growth in the vorticity observed in [11] is the first known result in this direction.

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